Trading Rules Over Fundamentals: A Stock Price Formula for High Frequency Trading, Bubbles and Crashes

Godfrey Cadogan *
Working Paper

Comments welcome

January 18, 2012

*Corresponding address: Information Technology in Finance, Institute for Innovation and Technology Management, Ted Rogers School of Management, Ryerson University, 575 Bay, Toronto, ON M5G 2C5; Tel: 786-324-6995; e-mail: godfrey.cadogan@ryerson.ca. I am grateful to Oliver Martin, Ali Akansu, Philip Obazee, John A. Cole, Christopher Faille, and Jan Kmenta for their comments on an earlier draft of the paper which improved its readability, and led to consideration of market microstructure effects. I thank Sam Cadogan for his research assistance. Research support from the Institute for Innovation and Technology Management is gratefully acknowledged. This version of the paper clarifies application of van der Corput's Lemma for oscillatory integrals to capital gains (in lieu of stock price) from [informed] trading, and includes new subsections on econometric specification of cost of carry for HFT, profitable trade strategies, and a case study of Nanex graphics for high frequency trading of natural gas (NG) index futures on the NYMEX. All errors which may remain are my own.

Abstract

In this paper we present a simple closed form stock price formula, which captures empirical regularities of high frequency trading (HFT), based on two factors: (1) exposure to hedge factor; and (2) hedge factor volatility. Thus, the parsimonious formula is not based on fundamental valuation. For applications, we first show that in tandem with a cost of carry model, it allows us to use exposure to and volatility of E-mini contracts to estimate dynamic hedge ratios, and mark-to-market capital gains on contracts. Second, we show that for given exposure to hedge factor, and suitable specification of hedge factor volatility, HFT stock price has a closed form double exponential representation. There, in periods of uncertainty, if volatility is above historic average, a relatively small short selling trade strategy is magnified exponentially, and the stock price plummets when such strategies are phased locked for a sufficient large number of traders. Third, we demonstrate how asymmetric response to news is incorporated in the stock price by and through an endogenous EGARCH type volatility process for past returns; and find that intraday returns have a U-shaped pattern inherited from HFT strategies. Fourth, we show that for any given sub-period, capital gains from trading is bounded from below (crash), i.e. flight to quality, but not from above (bubble), i.e. confidence, when phased locked trade strategies violate prerequisites of van der Corput's Lemma for oscillatory integrals. Fifth, we provide a taxonomy of trading strategies which reveal that high HFT Sharpe ratios, and profitability, rests on exposure to hedge factor, trading costs, volatility thresholds, and algorithm ability to predict volatility induced by bid-ask bounce or otherwise. Thus, extant regulatory proposals to control price dynamics of select stocks, i.e., pause rules such as "limit up/limit down" bands over 5-minute rolling windows, may mitigate but not stop future market crashes or price bubbles from manifesting in underlying indexes that exhibit HFT stock price dynamics.

Keywords: high frequency trading, hedge factor volatility, price reversal, market crash, price bubbles, fundamental valuation, van der Corput's Lemma, Sharpe ratio, cost of carry

JEL Classification Codes: C02, G11, G12, G13

Contents

List of Figures

1 Introduction

> ”A theory is a good theory if it satisfies two requirements: It must accurately describe a large class of observations on the basis of a model that contains only a few arbitrary elements, and it must make definite predictions about the results of future observations”. Stephen Hawkins, *A Brief History of Time*.

Recent studies indicate that high frequency trading (HFT) accounts for about 77% of trade volume in the UK[1] and upwards of 70% of trading volume in U.S. equity markets, and that ”[a]bout 80% of this trading is concentrated in 20% of the most liquid and popular stocks, commodities and/or currencies”[2]. Thus, at any given time an observed stock price in that universe reflects high frequency trading rules over price fundamentals[3]. In fact, a recent report by the Joint CFTC-SEC Advisory Committee summarizes emergent issues involving the impact of HFT on market microstructure and stock price dynamics[4]. This suggests the existence of stock price formulae for HFT, different from the class of Gordon[5] dividend based fundamental valuation[6] and present value models popularized in the litera-

[1](Sornette and Von der Becke, 2011, pg. 4)

[2]Whitten (2011). By definition, these statistics do not include dark pools or over the counter trades that are used to circumvent price impact on ”lit” exchanges dominated by HFT.

[3](Zhang, 2010, pg. 5) characterizes algorithmic trading rules thusly: ” HFT is a subset of algorithmic trading, or the use of computer programs for entering trading orders, with the computer algorithm deciding such aspects of the order as the timing, price, and order quantity. However, HFT distinguishes itself from general algorithmic trading in terms of holding periods and trading purposes”.

[4]See (Joint Advisory Committee on Emerging Regulatory Issues, 2011, pg. 2) (”The Committee believes that the September 30, 2010 Report of the CFTC and SEC Staffs to our Committee provides an excellent picture into the *new dynamics* of the electronic markets that now characterize trading in equity and related exchange traded derivatives”.) (emphasis added).

[5]Gordon (1959)

[6]See e.g., Tobin (1984). Compare (Campbell et al., 1997, Chapter 3.2) (bid-ask spreads due to market microstruc-

ture[7]. This paper provides such a formula by establishing stochastic equivalence between recent continuous time trade strategy and alpha representation theories introduced in Jarrow and Protter (2010), Jarrow (2010) and Cadogan (2011).

Evidently, high frequency traders quest for alpha, and their concomitant trade strategies are the driving forces behind short term stock price dynamics. In fact, a back of the envelope calculation by (Infantino and Itzhaki, 2010, pp. 10-11), using the Information Ratio (IR) metric popularized by (Grinold and Kahn, 2000, pg. 28), report that a high frequency trader who trades every 100 seconds can be 13,000 times less accurate than a star portfolio manager who trades about once per day, and still generate the same alpha. In that case, let $(\Omega, \{\mathscr{F}\}_{t\geq 0}, \mathbb{F}, P)$ be a filtered probability space, $S(t,\omega)$ be a realized stock price defined on that space, $X(t,\omega)$ be a realized hedge factor, $\beta_X(t,\omega)$ be a trade strategy factor, i.e. exposure to hedge factor, $\sigma_X(t,\omega)$ be hedge factor stochastic volatility, Q be a probability measure absolutely continous with respect to P, and $\tilde{B}_X(u,\omega)$ be a P-Brownian bridge or Q-Brownian motion that captures background driving stochastic process[8] for price reversal strategies[9]. We claim that the stock price representation

ture distorts fundamental price and cause serial correlation in transaction prices).

[7]See e.g., Shiller (1989), and Shiller and Beltratti (1992) who used annual returns, in a variant of Gordon's model of fundamental valuation for stock prices, and reject rational expectations present value models. But compare, (Madhavan, 2011, pp 4-5) who reports that "The futures market did not exhibit the extreme price movements seen in equities, which suggest that the Flash Crash [of May 6, 2010] might be related to the specific nature of the equity market structure".

[8]Christensen et al. (2011) report that granular tick by tick data reveal that jump variation account for at most 1% of price jumps, and that volatility drives ultra high frequency trading. Thus, Brownian motion without jumps is consistent with empirical regularities of HFT. We take notice of the importance of volatility for pricing futures contracts in an econometric specification

$$r_F(t) = -(i-d) + h(t)\left(\frac{\sigma_F^2(t)}{\sigma_S(t)}\right) + \Delta\eta(t)$$

for cost of carry, motivated by the formula, in subsection 4.1 on page 19, *infra*.

[9](Kirilenko et al., 2011, pg. 3) ("[N]et holdings of HFT fluctuate around zero")

for high frequency trading is given by

$$S(t,\omega) = S(t_0)\exp\Big(\int_{t_0}^{t} \sigma_X(u,,\omega)\beta_X(u,\omega)d\tilde{B}_X(u,\omega)\Big) \tag{1.1}$$

For instance, this formula allows us to use exposure (β_X) to and volatility (σ_X) of E-mini contracts (X) to predict movements in an underlying index (S)[10]. In fact, we show that the for a dyadic partition $t_j^{(n)},\ j=0,\cdots,2^n-1$, of the interval $[t_0,t]$, for risk free rate r_f, the HFT Sharpe ratio is given by

$$\begin{aligned} S_{ratio}(t,\omega) = &\left(\frac{\beta_X(t,\omega)\tilde{\varepsilon}_X(t,\omega)-r_f}{\sigma_X(t,\omega)}\right) \\ &\exp\left(\sum_{j=0}^{2^n-1}\overbrace{\sigma_X(t_j^{(n)},\omega)\beta_X(t_j^{(n)},\omega)\tilde{\varepsilon}_X(t_j^{(n)},\omega)}^{\text{past observations}}\right) \end{aligned} \tag{1.2}$$

where $\tilde{\varepsilon}_X$ corresponds to news[11] or market sentiment. For given exposure β_X if traders algorithmic forecasts about the direction of news (or market sentiment), i.e. $\operatorname{sgn}(\tilde{\varepsilon}_X(t,\omega))$ are accurate, then the Sharpe ratio will be high[12] when $\beta_X\tilde{\varepsilon}_X > r_f$ and

$$\sum_{j=0}^{2^n-1}\overbrace{\sigma_X(t_j^{(n)},\omega)\beta_X(t_j^{(n)},\omega)\tilde{\varepsilon}_X(t_j^{(n)},\omega)}^{\text{past observations}} > 0$$

because of the multiplicative exponential term that depends on past observations[13].

[10] This is indeed the case. See e.g. Kaminska (2011)

[11] See e.g. (Engle and Ng, 1993, pg. 1751) who interpret $\operatorname{sgn}(\tilde{\varepsilon}_X) > 0$ as good news, i.e., unexpected increase in price, and $\operatorname{sgn}(\tilde{\varepsilon}_X) < 0$ as bad news, i.e., unexpected decrease in price.

[12] (Brogaard, 2010, pg. 2) estimated a Sharpe ratio of 4.5 for HFT.

[13] See (Brogaard, 2010, pg. 2) ("I find past returns are important and so perform a logit regression analysis on past returns for different HFTs buying/selling and liquidity providing/demanding activities. The results suggest HFTs engage in a price reversal strategy. In addition, the results are strongest for past returns that are associated with a buyer-seller order imbalance.").

Suppose that ”X-factor” volatility has a continuous time GARCH(1,1) representation given by

$$d\sigma_X^2(t,\omega) = \theta(\eta - \sigma_X^2(t,\omega))dt + \xi\sigma_X^2(t,\omega)dW_{\sigma_X}(t,\omega) \tag{1.3}$$

where θ is the rate of reversion to mean volatility η, ξ is a scale parameter, and W_{σ_X} is a background driving Brownian motion for stochastic volatility σ_X. After applying Girsanov's Theorem to remove the drift in 1.3, and then substituting the resultant expression in 1.1, we get the double exponential local martingale representation of the stock price given by

$$\begin{aligned} &S(t,\omega) = \\ &S(t_0)\exp(\exp(-\tfrac{1}{2}\xi\tilde{W}_{\sigma_X}(t_0)))\exp\Big(\int_{t_0}^{t}\beta_X(u,\omega)\exp(\tfrac{1}{2}\xi\tilde{W}_{\sigma_X}(u,\omega))d\tilde{B}_X(u,\omega)\Big) \end{aligned} \tag{1.4}$$

Further details on derivation of the formula, and description of variables are presented in the sequel.

As indicated above, our parsimonious formula is closed form and it is based on two factors: (1) exposure to hedge factor, and (2) hedge factor volatility. The formula plainly shows that for given exposure, $\beta_X(u,\omega)$, to the ”X-factor”, if uncertainty in the market is such that volatility is above historic average, i.e., $\sigma_X^2(t,\omega) > \eta$ is relatively high, a portfolio manager might want to reduce exposure in order to stabilize the price of $S(t,\omega)$. However, if $\beta_X(u,\omega)$ is reduced to the point where it is negative, i.e. there is short selling, then the price of $S(t,\omega)$ will be in an exponentially downward spiral if volatility continues to increase[14]. Thus, feedback effects[15] between $S(t,\omega)$, the "X-factor" and exposure control or trade strategy $\beta_X(u,\omega)$ determines the price of S[16]. In this setup, so called market fundamentals do not determine the price. In fact, we show in Proposition 4.7 that HFT profitability rests on trader exposure to hedge factor, and ability to read market signals that portend accurate volatility forecasts.

One of the key results of our paper is the introduction of an admissible lower bound of zero (crash), and unbounded upper limit (bubble) for capital gains in high frequency trading environments. The analytics for that result are based on violation of van der Corput's Lemma for oscillatory integrals[17], and they produced a taxonomy of profitable trade strategies in the sequel. Recent proposals by the Joint Advisory Committee on Emerging Regulatory Issues include so called "Pause rules" extended to limit up\ limit down bands for the price of select stocks over rolling 5-minute windows[18]. Assuming without deciding that such recom-

[14] This prediction is supported by (Kirilenko et al., 2011, pg. 3) who observed:

> During the Flash Crash, High Frequency Traders initially bought contracts from Fundamental Sellers. After several minutes, HFTs proceeded to sell contracts and compete for liquidity with Fundamental Sellers. In this sense, the trading of HFTs, appears to have exacerbated the downward move in prices. In addition, HFTs appeared to rapidly buy and sell contracts from one another many times, generating a hot potato effect before Fundamental Buyers were attracted by the rapidly falling prices to step in and take these contracts off the market.

[15] See (Øksendal, 2003, pg. 237) for taxonomy of admissible [feedback] control functions. See also, de Long et al. (1990) (positive feedback trading by noise traders increase market volatility).

[16] For instance, (Madhavan, 2011, pg. 5) distinguishes between rules based algorithmic trading, and comparatively opaque high frequency trading in which traders play a signal jamming game of quote stuffing. There, large orders are posted and immediately canceled, i.e. reversed. See McTague (2010). This price reversal strategy is captured by $\beta_X(u,\omega)$ and embedded in our stock price.

[17] See e.g., (Stein, 1993, pg. 332).

[18] See e.g. (Joint Advisory Committee on Emerging Regulatory Issues, 2011, Recommendation #3, pg. 5)

mendations would mitigate order imbalances that may trigger excess volatility, its unclear how pause rules would affect the behaviour of an underlying basket of stocks or derivatives that may not be covered by the recommendation. In which case, an underlying stock index (comprised of stocks not covered by the recommendation) priced by our formula remains vulnerable to crashes and bubbles. The efficacy of our formula is supported by recent research on HFT which we review next.

An empirical study by Zhang (2010) found that high frequency trading (HFT) increases stock price volatility. In particular, ”the positive correlation between HFT and volatility is stronger when market uncertainty is high, a time when markets are especially vulnerable to aggressive to HFT”[19] A result predicted by our formula. Further, he used earnings surprise and analysts forecast as proxies for news and firm fundamentals to examine HFT response to price shocks. He found that ”the incremental price reaction associated with HFT are almost entirely reversed in the subsequent period”[20]. That finding is consistent with the price reversal strategy predicted by our theory. And the response to news is captured by our continuous time GARCH(1,1) specification[21] to hedge factor volatility incorporated in our closed form formula in an example presented in the sequel.

(Jarrow and Protter, 2011, pg. 2) presented a continuous time signalling model in which HFT ”trades can create increased volatility and mispricings” when

[19] (Zhang, 2010, pg. 3).
[20] *Ibid*.
[21] See e.g., (Engle, 2004, pg. 408).

high frequency traders observe a common signal. This is functionally equivalent to the trade strategy factor embedded in our closed form formula. However, those authors suggests that ”predatory aspects of high frequency trading” stem from the speed advantage of HFT, and that perhaps policy analysts should focus on that aspect. By contrast, our formula suggests that limit on short sales would mitigate the problem while still permitting the status quo.

A tangentially related paper[22] by Madhavan (2011) used a market segmentation theory, based on application of the Herfindahl Index, of market microstructure to explain the Flash Crash of May 6, 2010. Of relevance to us is (Madhavan, 2011, pg. 4) observation, that the recent joint SEC and CFTC report on the Flash Crash identified a trader's failure to set a price limit on a large E-mini futures contract, used to hedge an equity position, as the catalyst for the crash[23]. There, stock price movements were magnified by a feedback loop. An event predicted by, and embedded in, our stock price formula for high frequency trading as indicated in subsubsection 4.5.1. (Madhavan, 2011, pg. 6) also provides a taxonomy of high frequency trading strategies from papers he reviewed.

The rest of the paper proceeds as follows. In section 2 we introduce the model. In section 3 we derive the HFT stock price formula. The main result there is Proposition 3.3. In section 4, we briefly discuss the formula's implications for econometric estimation of optimal hedge ratio in cost of carry models in

[22]See also, (Sornette and Von der Becke, 2011, pp. 4-5) who argue that volume is not the same thing as liquidity, and that HFT reduces welfare of the real economy.

[23]Some analysts believe that algorithmic trading and or HFT is responsible for as much as a 10-fold decrease in market depth for S&P 500 E-mini contracts. See Kaminska (2011).

subsection 4.1. Next we apply the formula in the context of stochastic volatility in Lemma 4.1; characterize intraday return patterns in in Lemma 4.2; and the impact of phase locked high frequency trading strategies on capital gains in Proposition 4.6. Proposition 4.7 characterizes profitable HFT strategies. In subsubsection 4.5.1 we apply our theory to some graphics generated by Nanex to see if it explains observed trade phenomenon in natural gas index futures. In section 5 we conclude with perspectives on the implications of the formula for trade cycles in the context of quantum cognition.

2 The Model

We begin by stating the alpha representation theorems of Cadogan (2011); Jarrow and Protter (2010) in seriatim on the basis of the assumptions in Jarrow (2010). Then we show that the two theorems are stochastically equivalent–at least for a single factor–and provide some analytics from which the HFT stocp price formula is derived.

Assumption 2.1. *Asset markets are competitive and frictionless with continuous trading of a finite number of assets.*

Assumption 2.2. *Asset prices are adapted to a filtration of background driving Brownian motion.*

Assumption 2.3. *Prices are ex-dividend.*

Theorem 2.4 (Trading strategy representation. Cadogan (2011))**.**

Let $(\Omega, \mathscr{F}_t, \mathbb{F}, P)$ *be a filtered probability space, and* $Z = \{Z_s, \mathscr{F}_s;\ 0 \leq s < \infty\}$ *be a hedge factor matrix process on the augmented filtration* $\mathbb{F}$*. Furthermore, let* $a^{(i,k)}(Z_s)$ *be the* (i,k)*-th element in the expansion of the transformation matrix* $(Z_s^T Z_s)^{-1} Z_s^T$*, and* $B = \{B(s), \mathscr{F}_s;\ s \geq 0\}$ *be Brownian motion adapted to* $\mathbb{F}$ *such that* $B(0) = x$*. Assuming that* B *is the background driving Brownian motion for high frequency trading, the hedge factor sensitivity process, i.e. trading strategy,* $\gamma = \{\gamma_s, \mathscr{F}_s; 0 \leq s < \infty\}$ *generated by portfolio manager market timing for Brownian motion starting at the point* $x \geq 0$ *has representation*

$$d\gamma^{(i)}(t,\omega) = \sum_{k=1}^{j} a^{(i,k)}(Z_t) \left[\frac{x}{1-t} \right] dt - \sum_{k=1}^{j} a^{(i,k)}(Z_t) dB(t,\omega),\ x \geq 0$$

for the i-th hedge factor $i = 1, \ldots, p$*, and* $0 \leq t \leq 1$*.*

□

Proof. See (Cadogan, 2011, Thm. 4.6). □

Remark 2.1. (Cadogan, 2011, §2.2) presented a data mining algorithm based on martingale system equations to identify excess returns and describe high frequency trade strategy.

□

Cadogan (2011) computes an alpha vector $\boldsymbol{\alpha}(t,\omega) = Z\boldsymbol{\gamma}(t,\omega)$ where Z is a matrix of hedge factors and $\boldsymbol{\gamma}(t,\omega)$ is a vector of hedge factor sensitivity that constitutes the active portfolio manager's trading strategy. This parametrization is consistent with hedge factor models[24] such as Fama and French (1993).

Theorem 2.5 ((Jarrow and Protter (2010))).

Given no arbitrage, there exist K portfolio price processes $\{X_1(t),\ldots,X_K(t)\}$ such that an arbitrary security's excess return with respect to the default free spot interest rate, i.e. risk free rate, r_t can be written

$$\frac{dS_i(t)}{S_i(t)} - r_t dt = \sum_{k=1}^{K} \beta_{ik}(t)\left(\frac{dX_i(t)}{X_i(t)} - r_t dt\right) + \delta_i(t) d\varepsilon_i(t) \tag{2.1}$$

where $\varepsilon_i(t)$ is a Brownian motion under $(P,\mathscr{F}_t)$ independent of $X_k(t)$ for $k = 1,\ldots,K$ and $\delta_i(t) = \sum_{k=K+1}^{d} \sigma_{ik}^2(t)$

□

Proof. See (Jarrow and Protter, 2010, Thm. 4). □

In contrast to Cadogan (2011) endogenous alpha, (Jarrow, 2010, pg. 18) surmised that "[i]n active portfolio management, an econometrician would add a constant $\alpha_i(t)dt$ to [the right hand side of] this model". So Cadogan (2011) alpha is adaptive, whereas Jarrow (2010) alpha is exogenous. Without loss of generality we assume that the filtration in each model is with respect to background driving Brownian motion.

[24] See Noehel et al. (2010) for a review.

2.1 Trade strategy representation of alpha in single factor CAPM

If our trade strategy representation theory is well defined, then it should shed light on the behavior of *alpha* in a single factor model like CAPM where there is no hedge factor. In particular, for *cumulative alpha* $A(t)$ let[25]

$$A(t) = Z\boldsymbol{\gamma}(t) \tag{2.2}$$

where Z is a hedge factor matrix, and $\boldsymbol{\gamma}(t)$ is a trade strategy vector. Thus, for some vector $\boldsymbol{\alpha}(t)$ we have

$$d\{A(t)|Z\} = \boldsymbol{\alpha}(t)dt = Zd\boldsymbol{\gamma}(t). \text{ Let} \tag{2.3}$$

$$Z = \mathbb{1}_{\{n\}} \tag{2.4}$$

So that

$$(Z^T Z)^{-1} Z^T = n^{-1}\mathbb{1}^T_{\{n\}} \text{ and } a^{(1,k)}(Z_s) = n^{-1},\ k = 1,\dots,n \tag{2.5}$$

Substitution of these values in Theorem 2.4 gives us, by abuse of notation, the scalar equation for *trade strategy alpha*

$$-\alpha(t)dt = -d\gamma^{(1)}(t) = -\frac{x}{1-t}dt + dB(t) \tag{2.6}$$

[25] See (Christopherson et al., 1998, pp. 121-122) for representation of alpha conditioned on a Z-matrix of economic information.

That is the equation of a Brownian bridge starting at $B(0) = x$ on the interval $[0,1]$. See (Karlin and Taylor, 1981, pg. 268). So that trade strategy alpha is given by

$$d\gamma^{(1)}(t) = -dB^{br}(t) \tag{2.7}$$

$$\gamma^{(1)}(t) = B^{br}(0) - B^{br}(t) \tag{2.8}$$

However, there is more. According to Girsanov's formula in (Øksendal, 2003, pg. 162), we have an equivalent probability measure Q based on the martingale transform

$$M(t,\omega) = \exp\left(\int_0^t \frac{x}{1-s} dB(s,\omega) - \int_0^t \left(\frac{x}{1-s}\right)^2 ds\right) \tag{2.9}$$

$$dQ(\omega) = M(T,\omega)dP(\omega), \quad 0 \le t \le T \le 1 \tag{2.10}$$

Furthermore, we have the Q-Brownian motion

$$\hat{B}(t) = -\int_0^t \frac{x}{1-s} ds + B(t), \quad \text{and} \tag{2.11}$$

$$d\gamma^{(1)}(t) = -d\hat{B}(t) = -dB^{br}(t) \tag{2.12}$$

In other words, $\gamma^{(1)}$ is a Q-Brownian motion–in this case a Brownian bridge–that reverts to the origin starting at x. We note that

$$\frac{d\gamma^{(1)}(t)}{dt} = -\frac{dB^{br}(t)}{dt} = \tilde{\varepsilon}_t \tag{2.13}$$

Hence the "residual(s)" $\tilde{\varepsilon}_t$, associated with rate of change of Jensen's alpha, have an approximately skewed U-shape pattern on $[0,1]$. Additionallly, (Karlin and Taylor, 1981, pg. 270) show that the Brownian bridge can be represented by the Gaussian process

$$G(t) = (1-t)B\left(\frac{t}{1-t}\right) \tag{2.14}$$

$$E^Q[G(s_1)G(s_2)] = \begin{cases} s_1(1-s_2) & s_1 < s_2 \\ \\ s_2(1-s_1) & s_2 < s_1 \end{cases} \tag{2.15}$$

This is the so called *Doob transformation*, see (Doob, 1949, pp. 394-395), and

$$B(t) = (1+t)B\left(\frac{t}{1+t}\right) \tag{2.16}$$

(Karatzas and Shreve, 1991, pg. 359) also provide further analytics which show that we can write the trade strategy alpha as

$$d\gamma^{(1)}(t) = \frac{1-\gamma^{(1)}}{1-t}dt + dB(t); \quad 0 \leq t \leq 1, \;\; \gamma^{(1)} = 0 \tag{2.17}$$

$$M(t) = \int_0^t \frac{dB(s)}{1-s} \tag{2.18}$$

$$T(s) = \inf\{t| < M >_t > s\} \tag{2.19}$$

$$G(t) = B_{<M>_{T(t)}} \tag{2.20}$$

where $< M >_t$ is the quadratic variation[26] of the local martingale M.

[26] See (Karatzas and Shreve, 1991, pg. 31).

3 High Frequency Trading Stock Price Formula

We use a single factor representation of Theorem 2.4 and Theorem 2.5 with $K = 1$ to derive the stock price formula as follows[27]. Using (Jarrow, 2010, eq(5), pg. 18) formulation, let

$$\alpha(t)dt = \frac{dS(t)}{S(t)} - r_t dt - \beta_1(t)\left(\frac{dX_1(t)}{X_1(t)} - r_t dt\right) - \sigma dB(t) \tag{3.1}$$

$$= \frac{dS(t)}{S(t)} - \beta_1(t)\frac{dX_1(t)}{X_1(t)} - r_t(1-\beta_1(t))dt - \sigma dB(t) \tag{3.2}$$

Comparison with 2.6 suggests that the two alphas are equivalent if the following identifying restrictions are imposed

$$d\gamma^{(1)}(t) \equiv \alpha(t)dt \tag{3.3}$$

$$\Rightarrow \frac{dS(t)}{S(t)} - \beta_1(t)\frac{dX_1(t)}{X_1(t)} = 0 \tag{3.4}$$

$$\sigma = 1 \tag{3.5}$$

$$\frac{x}{1-t} = (\beta_1(t) - 1)r_t \tag{3.6}$$

So that

$$-d\gamma^{(1)} = -(\beta_1(t) - 1)r_t dt + dB(t) \tag{3.7}$$

[27](Jarrow and Protter, 2010, pg. 12, eq. (12)) refer to the ensuing as a 'regression equation" for which an econometrican tests the null hypothesis $H_0 : \alpha(t)dt = 0$. However, (Cadogan, 2011, eq. 2.1) applied asymptotic theory to econometric specification of a canonical multifactor linear asset pricing model augmented with portfolio manager trading strategy to identify portfolio alpha.

In fact, if for some constant drift μ_X, volatility σ_X, and P-Brownian motion B_X we specify the "hedge factor" dynamics

$$\frac{dX_1(t)}{X_1(t)} = \mu_X dt + \sigma_X dB_X(t) \tag{3.8}$$

then after applying Girsanov's change of measure to 3.8, and by abuse of notation [and without loss of generality] setting $\beta_X(t,\omega) = \beta_1(t)$, we can rewrite 3.4 as

$$\frac{dS(t)}{S(t)} = \beta_X(t,\omega)\sigma_X d\tilde{B}_X(t) \tag{3.9}$$

for some Q-Brownian motion $\tilde{B}_X$. These restrictions, required for functional equivalence between the two models, are admissible and fairly mild. We summarize the forgoing in the following

Proposition 3.1 (Stochastic equivalence of alpha in single factor models.)**.** *Assume that asset prices are determined by a single factor linear asst pricing model. Then the Jarrow and Protter (2010) return model is stochastically equivalent to Cadogan (2011) trading strategy representation model.*

$\square$

Corollary 3.2 (Jarrow (2010) trade strategy drift factor)**.**
$\{\beta_1(t)\}_{t\in[0,T]}$ *in Jarrow (2010) is a trade strategy [drift] factor.*

$\square$

Equation 3.9 plainly shows that the closed form expression for the stock price over some interval $[t_0, t]$ is now:

$$d\{\ln(S(t,\omega))\} = \beta_X(t,\omega)\sigma_X d\tilde{B}_X(t,\omega) \tag{3.10}$$

$$\Rightarrow \int_{t_0}^{t} d\{\ln(S(t,\omega))\} = \sigma_X \int_{t_0}^{t} \beta_X(u,\omega) d\tilde{B}_X(u,\omega) \tag{3.11}$$

$$\Rightarrow S(t,\omega) = S(t_0)\exp\Big(\int_{t_0}^{t} \underbrace{\sigma_X}_{\text{volatility}} \underbrace{\beta_X(u,\omega)}_{\text{exposure}} \underbrace{d\tilde{B}_X(u,\omega)}_{\text{news}}\Big) \tag{3.12}$$

This formula allows us to use E-mini futures contracts to predict movements in an underlying index. For constant volatility σ_X, this gives us the following

Proposition 3.3 (High Frequency Trading Stock Price Formula)**.**
Let $(\Omega, \mathscr{F}, \mathbb{F}, P)$ be a probability space with augmented filtration with respect to Brownian motion $B = \{B(t,\omega); 0 \le t < \infty\}$. Let S be a stock price, and X be a hedge factor with volatility σ_X, adapted to the filtration; and β_X be the exposure of S to X. Then the stock price over the interval $[t_0, t]$ is given by

$$S(t,\omega) = S(t_0)\exp\Big(\int_{t_0}^{t} \sigma_X \beta_X(u,\omega) d\tilde{B}_X(u,\omega)\Big)$$

□

4 Applications

We provide five applications of our stock price representation theory in seriatim in the sequel. First, we consider the implications of our formula for cost of carry models, including but not limited to estimation of dynamic hedge ratios. Second, we consider the case of stochastic volatility when σ_X has continuous time GARCH(1,1) representation[28]. We use Girsanov's Theorem to extend the formula to the case of double exponential representation of HFT stock price. correlated background driving processes. Third, we consider the case of intraday return patterns generated by our formula, and show how it has EGARCH features. There, we show how news is incorporated in the HFT stock price. And we provide a Sharpe ratio estimator for HFT. Fourth, we present bounds of capital gains arising from phase locked processes. There, we use van der Corput's Lemma[29] for phase functions to motivate our result. Fifth, we characterize cost and volatility structures, and derive profitable trade strategies for HFT. Whereupon we close with application of the formula to a case study of Nanex graphics for high frequency trading of natural gas index futures on NYMEX.

4.1 Implications for cost of carry models and optimal hedge ratio

Proposition 3.3 has implications for the cost of carry model popularized in pricing futures. Specifically, let i and d be short term interest rate and dividend yield for stocks in an index $S(t)$ over a given horizon $[0,T]$ for a hedge factor, i.e. futures price, $F(t,T)$ at time t. The cost of carry model posits that at time $t \in [0,T]$ the spot price $S(t)$ of the index is $S(t) = F(t,T)\exp(-(i-d)(T-t))$. By equating

[28] In practice we would use the implied volatility of the "X-factor". Furthermore, there are several different specifications for stochastic volatility popularized in the literature. See Shephard (1996) for a review. However, "[t]he GARCH(1,1) specification is the workhorse of financial applications," (Engle, 2004, pg. 408).

[29] See (Stein, 1993, Prop. 2, pg. 332)

this to our formula we find

$$F(t,T)\exp(-(i-d)(T-t)) = S(0)\exp\Big(\int_0^t \sigma_F(u,\omega)\beta_F(u,\omega)d\tilde{B}_F(u,\omega)\Big) \tag{4.1}$$

$$\beta_F(u,\omega) = \frac{h(u,\omega)\sigma_F(u,\omega)}{\sigma_S(u,\omega)} \tag{4.2}$$

where $\sigma_S(u,\omega)$ is volatility of spot index and $h(u,\omega) = \frac{N_F(u,\omega)}{N_S(u,\omega)}$ is a *hedge ratio*, i.e. Markov control variable, for amount of futures contracts to stock index, as indicated in (Hull, 2006, pg. 73). After taking logs of both sides of the incipient equation, and discretizing the stochastic integral term we get the following admissible parametrization

$$\ln(F(t,T)) = \ln(S(0)) + (i-d)(T-t) + \sum_{k=1}^{t} \beta_F(k)\sigma_F(k) + \eta(t) \tag{4.3}$$

During a time interval $[s^+,t]$, the mark-to-market capital gains of the futures contract is given by

$$\ln(F(t,T)) - \ln(F(s^+,T)) = -(i-d)(t-s^+) + \sum_{k=s^+}^{t} \beta_F(k)\sigma_F(k) + \eta(t) - \eta(s^+) \tag{4.4}$$

$$= -(i-d)(t-s^+) + \sum_{k=s^+}^{t} h(k)\frac{\sigma_F^2(k)}{\sigma_S(k)} + \eta(t) - \eta(s^+) \tag{4.5}$$

By setting $s = t - 1$, where "1" is an appropriate time scale in milliseconds, we get the returns

$$\ln(F(t,T)) - \ln(F(t-1,T)) = r_F(t) = -(i-d) + \beta_F(t)\sigma_F(t) + \Delta\eta(t) \tag{4.6}$$

$$\Rightarrow r_F(t) = -(i-d) + h(t)\left(\frac{\sigma_F^2(t)}{\sigma_S(t)}\right) + \varepsilon(t) \tag{4.7}$$

where r_F is log-returns on futures, $\sigma_F(t)$ and $\sigma_S(t)$ have GARCH type dynamics, η is assumed stationary so that $\varepsilon(t) = \Delta\eta(t)$ is a "news" term, and $h(t)$ is a rough estimate of time varying optimal hedge ratio. We use s^+ to highlight the fact that the unit of measurement is in milliseconds. According to results reported in Christensen et al. (2011) the specification above is consistent with the notion that ultra-HFT is driven by volatility with negligible price jumps, and that traders get in and out of positions quite rapidly to capture mark-to-market capital gains in (4.4). For example, $0 < \varepsilon > i - d$ is "good news"–the asset is relatively under priced. Whereas $0 > \varepsilon < (i - d)$ is 'bad news"–the asset is relatively overpriced. Price volatility may be generated by HFT gaming the bid-ask spread through order flow to capitalize on bid-ask bounce, since transaction prices tend to be higher at the bid and lower at the sell[30]. In other words, a marketmaker may be subject to quote stuffing to extract a desired price. If a trader has high latency trading technology and [s]he is able to predict the direction of prices between bid-ask spreads, then [s]he can make a miniscule profit on each trade[31]. Multiplied by

[30] Aldridge (2011) demonstrated that that is easier said than done.

[31] See (Campbell et al., 1997, Chapter 3.2) for details on inner workings of market microstructure.

millions of trades, that number can be significant. Moreover, it is consistent with price reversal strategies predicated on mean reverting stationary processes $\eta(t)$ which may be due to serial correlation induced by bid-ask spread. In fact, (4.4) answers the question about derivation of the price signal ε posed by Avellaneda and Lee (2010) relative value pricing equation (3). In order not to overload the paper we do not address the independently important econometric issues arising from that specification of futures prices. Cf. MacKinlay and Ramaswamy (1988); and Stoll and Whaley (1990).

4.2 The impact of stochastic volatility on HFT stock prices

This specification of the model permits extension to representation(s) of stochastic volatility for the "X-factor" to mitigate potential simultaneity bias. For exposition, and analytic tractability we specify a GARCH(1,1) representation for σ_X.

$$d\sigma_X^2(t,\omega) = \theta(\eta - \sigma_X^2(t,\omega))dt + \xi\sigma_X^2(t,\omega)dW_{\sigma_X}(t,\omega) \tag{4.8}$$

where θ is the rate of reversion to mean volatility η, ξ is a scale parameter, and W_{σ_X} is a background driving Brownian motion. Girsanov change of measure formula, see e.g. (Øksendal, 2003, Thm. II, pg. 164), posits the existence of a Q-Brownian motion $\tilde{W}_{\sigma_X}(t,\omega)$ and local martingale representation

$$d\sigma_X^2(t,\omega) = \xi\sigma_X^2(t,\omega)d\tilde{W}_{\sigma_X}(t,\omega) \tag{4.9}$$

$$\Rightarrow \sigma_X^2(t,\omega) = \exp\Big(\xi\int_{t_0}^{t} d\tilde{W}_{\sigma_X}(u)\Big) \tag{4.10}$$

$$\Rightarrow \sigma_X(t,\omega) = \exp\Big(\tfrac{1}{2}\xi(\tilde{W}_{\sigma_X}(u) - \tilde{W}_{\sigma_X}(t_0))\Big) \tag{4.11}$$

Substitution in 3.12 gives us the double exponential representation

$$S(t,\omega) = S(t_0)\exp\Big(\int_{t_0}^{t}\exp(\tfrac{1}{2}\xi(\tilde{W}_{\sigma_X}(u,\omega) - \tilde{W}_{\sigma_X}(t_0)))\beta_X(u,\omega)d\tilde{B}_X(u,\omega)\Big) \tag{4.12}$$

$$\begin{aligned} &= S(t_0)\exp(\exp(-\tfrac{1}{2}\xi\tilde{W}_{\sigma_X}(t_0))) \\ &\exp\Big(\int_{t_0}^{t}\beta_X(u,\omega)\exp(\tfrac{1}{2}\xi\tilde{W}_{\sigma_X}(u,\omega))d\tilde{B}_X(u,\omega)\Big) \end{aligned} \tag{4.13}$$

The formula plainly shows that for given stock price exposure, β_X, to the "X-factor", if uncertainty in the market is such that volatility is above historic average, i.e., $\sigma_X^2(t,\omega) > \eta$ because uncertainty about the news $\tilde{W}_{\sigma_X}$ is relatively high, a portfolio manager might want to reduce exposure in order to stabilize the price of S. However, if β_X is reduced to the point where it is negative, i.e. there is short selling, then the price of S will be in an exponentially downward spiral if volatility continues to increase. In the face of monotone phase locked strategies presented in subsection 4.4 this leads to a market crash of the stock price. Thus, feedback effects between S, the "X-factor" and exposure control or trade strategy β_X determines the price of S. In this setup, so called market fundamentals do not determine the price. We present this representation in the following

Lemma 4.1 (Double exponential HFT stock price representation)**.**
When stochastic volatility follows a continuous time GARCH(1,1) process, the high frequency trading stock price has a double exponential representation given in 4.13.

□

4.3 Implications for intraday return patterns and volatility

In this subsection we derive the corresponding formula for intraday returns $r_{HFT}(t,\omega)$ for HFT stock price formula. Assuming that σ_X is stochastic in Proposition 3.3 we have

$$\ln(S(t,\omega)) = \ln(S(t_0)) + \int_{t_0}^{t} \sigma_X(u,\omega)\beta_X(u,\omega)d\tilde{B}_X(u,\omega) \quad (4.14)$$

Consider the dyadic partition $\prod^{(n)}$ of $[t_0, t]$ such that

$$t_j^{(n)} = t_0 + j.2^{-n}(t - t_0) \tag{4.15}$$

$$d\ln(S(t,\omega)) = \frac{dS(t,\omega)}{S(t,\omega)}dt = \frac{\sigma_X(t,\omega)\beta_X(t,\omega)d\tilde{B}_X(t,\omega)}{S(t_0)\exp\left(\int_{t_0}^{t}\sigma_X(u,\omega)\beta_X(u,\omega)d\tilde{B}(u,\omega)\right)} \tag{4.16}$$

The discretized version of that equation, where we write $\frac{d\tilde{B}_X(t,\omega)}{dt} \approx \tilde{\varepsilon}_X(t,\omega)$, suggests that HFT intraday returns is given by

$$\begin{aligned} & r_{HFT}(t,\omega) = \\ & \sigma_X(t,\omega)\beta_X(t,\omega)\tilde{\varepsilon}_X(t,\omega)S(t_0)^{-1} \\ & \exp\left(-\sum_{j=0}^{2^n-1}\underbrace{\sigma_X(t_j^{(n)},\omega)\beta_X(t_j^{(n)},\omega)\tilde{\varepsilon}_X(t_j^{(n)},\omega)}_{\text{past observations}}\right) \end{aligned} \tag{4.17}$$

Examination of the latter equation shows that it inherits the U-shaped pattern from 2.13 by and through $\tilde{\varepsilon}_X(t,\omega)$. This theoretical U-shape result is supported by Admati and Pfleiderer (1988) in the context of intraday trade volume and optimal decisions of liquidity traders and informed traders. Moreover, assuming $S(t_0) = 1$, an admissible decomposition of HFT returns in the context of an ARCH specification is:

$$r_{HFT}(t,\omega) = \sigma_{HFT}(t,\omega)\tilde{\varepsilon}_{HFT}(t,\omega), \text{ where} \tag{4.18}$$

$$\sigma_{HFT}(t,\omega) = \sigma_X(t,\omega)\exp\Big(-\sum_{j=0}^{2^n-1}\sigma_X(t_j^{(n)},\omega)\beta_X(t_j^{(n)},\omega)\tilde{\varepsilon}_X(t_j^{(n)},\omega)\Big) \tag{4.19}$$

$$\tilde{\varepsilon}_{HFT}(t,\omega) = \beta_X(t,\omega)\tilde{\varepsilon}_X(t,\omega) \tag{4.20}$$

The Sharpe ratio for HFT returns for given risk free rate r_f is given by

$$S_{ratio}(t,\omega) = \frac{r_{HFT}(t,\omega)-r_f}{\sigma_{HFT}(t,\omega)} \tag{4.21}$$

$$= \frac{\beta_X(t,\omega)\tilde{\varepsilon}_X(t,\omega)}{\sigma_{HFT}(t,\omega)} - \frac{r_f}{\sigma_{HFT}(t,\omega)} \tag{4.22}$$

$$= \left(\frac{\beta_X(t,\omega)\tilde{\varepsilon}_X(t,\omega)-r_f}{\sigma_X(t,\omega)}\right)\exp\left(\sum_{j=0}^{2^n-1}\sigma_X(t_j^{(n)},\omega)\beta_X(t_j^{(n)},\omega)\tilde{\varepsilon}_X(t_j^{(n)},\omega)\right) \tag{4.23}$$

Examination of 4.19 shows that it admits an asymmetric response to news $\tilde{\varepsilon}_X(t_j^{(n)},\omega)$ about and exposure to the hedge factor X. This is functionally equivalent to Nelson (1991) EGARCH specification. And it may help explain why Busse (1999) found weak evidence of volatility timing in daily returns for active portfolio management in a sample of mutual funds based on his EGARCH specification. Equation 4.20 shows that high frequency traders response $\tilde{\varepsilon}_{HFT}(t,\omega)$ to news $\tilde{\varepsilon}_X(t,\omega)$ about the "X-factor" is controlled by stock price exposure $\beta_X(t,\omega)$. Also, for given HFT volatility, (4.23) shows that Sharpe ratio depends on exposure to hedge factor, and ability to interpret $\tilde{\varepsilon}_X(t,\omega)$, i.e., predict $\text{sgn}(\tilde{\varepsilon}_X(t,\omega))$, since r_f is relatively constant over short periods. This is tantamount to predicting the direction of $\tilde{\varepsilon}_X(t,\omega)$ by and through market sentiment, which lends itself to algorithmic trading from data mining. We summarize this result in the following

Lemma 4.2 (Intraday HFT return behaviour)**.**

The volatility of HFT returns depends on contemporaneous hedge factor volatility and asymmetrically on historic *news about and exposure to hedge factors. Intraday HFT returns exhibit EGARCH features, and inherit U-shaped patterns from price reversal strategies of high frequency traders.*

□

Lemma 4.3 (HFT Sharpe ratios)**.**

HFT Sharpe ratio in (4.23) *depends on trader ability to forecast the direction of hedge factor prices with algorithmic trading and data mining.*

□

We next extend the analysis to correlation between the background driving processes for hedge factor and hedge factor volatility.

Let the X-factor be a forward rate. The stochastic alpha beta rho (SABR) model[32] for X-dynamics posits:

$$dX(t,\omega) = \sigma_X(t,\omega)X^{\beta}(t,\omega)dB_X(t,\omega), \;\; 0 \leq \beta \leq 1 \tag{4.24}$$

$$d\sigma_X(t,\omega) = \alpha\sigma_X(t,\omega)dW_{\sigma_X}(t,\omega), \;\; \alpha \geq 0 \tag{4.25}$$

$$B_X(t,\omega) = \rho W_{\sigma_X}(t,\omega), \;\; |\rho| < 1 \tag{4.26}$$

where B_X and W_{σ_X} are background driving Brownian motions as indicated. Assuming without deciding that the SABR model is the correct one to specify the dynamics of X, the double exponential representation in 4.13 plainly shows that it includes the background driving Brownian motions in 4.26. In which case we extend the representation to

$$\begin{aligned} &S(t,\omega) = \\ &= S(t_0)\exp(\exp(-\tfrac{1}{2}\xi\tilde{W}_{\sigma_X}(t_0))) \\ &\exp\left(\rho\int_{t_0}^{t}\beta_X(u,\omega)\exp(\tfrac{1}{2}\xi\tilde{W}_{\sigma_X}(u,\omega))d\tilde{W}_{\sigma_X}(u,\omega)\right) \end{aligned} \tag{4.27}$$

That representation plainly shows that now the stock price depends on background driving dynamics of hedge factor volatility, and its correlation with underlying hedge factor dynamics. This suggests that for applications the implied volatility of hedge factor and or some variant of the VIX volatility index should be factored in HFT stock price formulae. Here again, the nature of the correlation coefficient ρ, i.e. whether its positive or negative for given exposure β_X, determines how the

[32]See Zhang (2011) for a recent review.

stock price responds to short selling. We close with the following

Lemma 4.4 (HFT stock price as a function of background driving processes)**.** *HFT stock price dynamics depends on the background driving process for hedge factor stochastic volatility, and its correlation with the background driving process for hedge factor dynamics.*

Remark 4.1. This lemma implies the existence of a Hidden Markov Model to capture the latent dynamics in hight frequency trading. In order not to overload the paper we will not address that issue.

□

4.4 van der Corput's lemma and phase locked capital gains

Before we examine the impact of phased locked trade strategies on HFT stock prices, we need the following

Proposition 4.5 (van der Corput's Lemma)**.** *(Stein, 1993, pg. 332)*
Suppose ϕ is real valued and smooth in (a,b), and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a,b)$. Then $|\int_a^b \exp(i\lambda\phi(x))dx| \leq c_k \lambda^{-\frac{1}{k}}$ holds when:

i. $k \geq 2$, or

ii. $k = 1$ and $\phi'(x)$ is monotonic

The bounds c_k is independent of ϕ and λ.

□

Let

$$\phi^j(t,\omega) = \int_{t_0}^{t} \beta_X^j(u,\omega)\,d\tilde{B}_X(u) \tag{4.28}$$

be a local martingale for the j-th high frequency trader. Furthermore, consider the complex valued stock price function

$$\hat{S}(t,\omega) = S(t_0)\exp\Big(i\sigma_X\phi^j(t)\Big) \tag{4.29}$$

For simplicity, let

$$S(t_0) = 1 \tag{4.30}$$

Define the oscillatory integrals over some interval $t_0 < a < t < b$ for the stochastic phase function $\phi^j(t,\omega)$

$$I^j(\sigma_X,\omega) = \int_a^b S(t,\omega)\,dt = \int_a^b \exp\Big(\sigma_X\phi^j(t,\omega)\Big)\,dt \tag{4.31}$$

$$\hat{I}^j(\sigma_X,\omega) = \int_a^b \hat{S}(t,\omega)\,dt = \int_a^b \exp\Big(i\sigma_X\phi^j(t,\omega)\Big)\,dt \tag{4.32}$$

Here, $I^j(\sigma_X,\omega)$ and $\hat{I}^j(\sigma_X,\omega)$ can be interpreted as capital gains over the interval $a < t < b$. Consider the differential

$$\phi^{j\prime}(t,\omega) = \beta_X^j(t,\omega)\,d\tilde{B}_X(t) \tag{4.33}$$

Let $\beta_X^j(t,\omega)$ be a monotone trade strategy over the trading horizon under consideration. Without loss of generality assume that $\tilde{B}_X$ is Q-Brownian motion, so that

$$|\phi^{j\prime}(t,\omega)| \geq 1 \Rightarrow |\phi^{j\prime}(t,\omega)|^2 \Rightarrow \beta^{j^2}(t,\omega)dt \geq 1 \tag{4.34}$$

These are the restrictions that must be imposed on trade strategy for Proposition 4.5 to hold. However, the proposition also suggests that if ϕ^j is real valued and smooth, i.e. differentiable, and $\phi^{j\prime}$ is monotonic over the interval (a,b), then the amplitude of the oscillatory integral has estimate

$$|\hat{I}^j(\sigma_X,\omega)| \leq c_k \sigma_X^{-\frac{1}{k}} \tag{4.35}$$

From the outset we note that when $|\phi^{j\prime}| < 1$, the conditions of Proposition 4.5 are violated so that

$$\hat{I}^j(\sigma_X,\omega) > c_k \sigma_X^{-\frac{1}{k}} \tag{4.36}$$

Furthermore, the conditions above are violated because $\phi^j(t,\omega)$ is a Brownian functional which oscillates rapidly near 0 so it is not differentiable there[33]. Specifically, $\phi^j(t,\omega)$ is a function of trade strategy, i.e., exposure to hedge factor, which fluctuates very rapidly near zero because HFT gets in and out of hedge positions

[33]The "0" can be translated to another time t_0 say and the rapid oscillation holds by virtue of the strong Markov property of Brownian motion. See (Karatzas and Shreve, 1991, pg. 86). For empirical evidence see (Kirilenko et al., 2011, pg. 3) ("We find that on May 6, the 16 trading accounts that we classify as HFTs traded over 1,455,000 contracts, accounting for almost a third of total trading volume on that day. Yet, net holdings of HFTs *fluctuated around zero so rapidly* that they rarely held more than 3,000 contracts long or short on that day").

very rapidly. The oscillatory integral $\hat{I}^j(\sigma_X, \omega)$ inherits the rapid fluctuations which are magnified exponentially. Thus, in a market with a finite number of traders N, say, we have

$$\sum_{j=1}^{N} |\hat{I}^j(\sigma_X, \omega)| \geq N c_k \sigma_X^{-\frac{1}{k}} \tag{4.37}$$

From 4.31, 2.6 and 4.36 we have

$$|I^j(\sigma_X, \omega)| \geq \hat{I}^j(\sigma_X) \tag{4.38}$$

$$\Rightarrow \sum_{j=1}^{N} |I^j(\sigma_X, \omega)| \geq \sum_{j=1}^{N} |\hat{I}^j(\sigma_X)| > N c_k \sigma_X^{-\frac{1}{k}} \tag{4.39}$$

$$\Rightarrow \bar{I}_N(\sigma_X) = \frac{1}{N} \sum_{j=1}^{N} |I^j(\sigma_X, \omega)| > c_k \sigma_X^{-\frac{1}{k}} \tag{4.40}$$

Equation 4.31 represents the impact of the j-th trader's strategy, β_X^j, on capital gains $I^j(\sigma_X, \omega)$. However, 4.39 shows that in the best case when volatility grows no slower than $\mathcal{O}(N)$, the cumulative effect of high frequency traders strategies $\{\beta_X^1, \ldots, \beta_X^N\}$ is unbounded from above. Moreover, 4.40 represents the phased locked, or average trade strategy effect on capital gains, which is also unbounded from above. Thus we proved the following

Proposition 4.6 (Phased locked stock price)**.**

Let $\{\beta_X^1, \ldots, \beta_X^N\}$ be the distribution of high frequency traders [monotone] strategies over a given trading horizon in a market with N traders. Let $I^j(\sigma_X, \omega)$ in 4.31 *be the capital gain impact of the j-th trade strategy induced by the stock price*

$$S(t,\omega) = S(t_0)\exp\Big(\int_{t_0}^{t} \sigma_X \beta_X^j(u,\omega)d\tilde{B}_X(u,\omega)\Big)$$

Let

$$\bar{I}_N(\sigma_X,\omega) = \frac{1}{N}\sum_{j=1}^{N} |I^j(\sigma_X,\omega)| > c_k \sigma_X^{-\frac{1}{k}}$$

be the average capital gain induced by phase locked high frequency trading strategies. Then capital gains are bounded from below, but not from above. In which case, the lower bound constitutes capital gains when the market crashes at $c_k = 0$*. When* $c_k > 0$ *capital gains are unbounded from above, and determined by high frequency trading strategies with admissible price bubbles.*

□

Remark 4.2. The path characteristics of the Brownian functional $\phi^j(t,\omega)$ are such that $c_k = 0$ corresponds to very rapid fluctuations, i.e. large jumps, in $\phi^{j'}(t,\omega)$ in 4.33 at or near t. In other words, volatility in the stock is extremely high and there aren't enough buyers for the phased locked short sales strategies. According to Kaminska (2011) arbitrageurs, and some high frequency traders withdraw from the market in the face of this kind of uncertainty. Thus, markets breakdown as there is a flight to quality and or liquidity in the sense of Akerlof (1970). Some analysts believe that to be the case in the Flash Crash of May 6, 2010. For "regular" $\phi^{j'}(t,\omega)$, i.e. there is more confidence and markets are bullish, $c_k > 0$ and we get potential price bubbles from phase locked strategies.

Remark 4.3. Technically, 4.40 is bounded from above by $N(b-a)$ when $\phi^{j}(t,\omega) = 0$. However, this corresponds to $\beta_X(t,\omega) = 0$ and $\phi^{j'}(t,\omega)$ is undefined at 0, i.e. jumps in $\phi^{j'}(t,\omega)$ are infinitely large. In effect, the upper bound is illusionary because the conditions of Proposition 4.5 are still violated.

4.5 Profitable HFT trade strategies

Let $C(\sigma_X)$ be the cost per trade. Assume that the average trader makes M trades. So that the average profit for the N traders be given by

$$\bar{\Pi}_N(\sigma_X,\omega) = \bar{I}_N(\sigma_X,\omega) - MC(\sigma_X) \tag{4.41}$$

The average profit per trade is given by

$$\bar{\Pi}_N^M(\sigma_X) = \bar{I}_N^M(\sigma_X,\omega) - C(\sigma_X), \text{ where} \tag{4.42}$$

$$\bar{I}_N^M(\sigma_X,\omega) = \frac{\bar{I}_N(\sigma_X,\omega)}{M} \tag{4.43}$$

This profit is characterized by the behaviour of the phase function $\phi^{j}(t,\omega)$ in 4.28. To see that, for given volatility σ_X, let

$$\mathrm{sgn}(d\tilde{B}_X(t,\omega)) > 0 \tag{4.44a}$$

$$\mathrm{sgn}(d\tilde{B}_X(t,\omega)) < 0 \tag{4.44b}$$

Christensen et al. (2011) report that examination of ultra high frequency data reveal that jump variation accounts for about 1% of price movements for ultra high frequency trading (UHFT). So that volatility, not price jumps, drives UHFT. In which case (4.44a) and (4.44b) represent "bullish" and "bearish" uncertainty, respectively. In that milieu, the average trade reports a profit when

$$\bar{\Pi}_N^M(\sigma_X,\omega) > 0 \Rightarrow \bar{I}_N^M(\sigma_X,\omega) > \max\{c_k\sigma_X^{-\frac{1}{k}},\, C(\sigma_X)\} \tag{4.45a}$$

$$\Rightarrow \beta_X^j(t,\omega) < 0 \text{ and } \operatorname{sgn}(d\tilde{B}_X(t,\omega)) < 0, \text{ or} \tag{4.45b}$$

$$\beta_X^j(t,\omega) > 0 \text{ and } \operatorname{sgn}(d\tilde{B}_X(t,\omega)) > 0 \tag{4.45c}$$

Equation 4.37 and (4.32) imply that[34] for

$$\Delta t = b - a \tag{4.46}$$

$$c_k\sigma_X^{-\frac{1}{k}} \leq S(t_0)\Delta t \tag{4.47}$$

Equating the terms in the maximand in (4.45a), and substituting in the equation above gives us the cost structure

$$\frac{C(\sigma_X)}{S(t_0)} \leq \Delta t \tag{4.48}$$

$$\Rightarrow \sigma_X \geq \left(\frac{c_k}{S(t_0)\Delta t}\right)^k \tag{4.49}$$

where k is the order of differentiability of the phase function. Since the latency of high frequency trades is in the order of milliseconds, the time interval Δt is quite small. So (4.48) suggests that the cost per trade must be extremely low in comparison to the price of the stock at the beginning of the trading period. Thus,

[34] Recall that we set $S(t_0) = 1$. It is being reintroduced here for expository purposes.

profitable trade strategies include short sell when volatility signals (market sentiment) bearish market in (4.45b), buy and hold when volatility signals bullish market in (4.45c), and trading cost per unit of time and volatility thresholds are as in (4.48) and (4.49), respectively. As the number of trades increase, i.e. $M\uparrow$, the average profit per trade decreases, i.e. $\bar{\Pi}^M_N(\sigma_X,\omega)\downarrow$. In practice, given the narrowness of bid ask spreads, and the serial correlation in price changes attributable to those spreads, a trader need only predict volatility caused by the bid-ask bounce. Given the dependence of cost and profits on volatility, and narrow spreads, the trade strategy above is consistent with the observation that high frequency traders make fractions of a penny on an intra-spread dollar[35], but execute such a large volume of trades that they are able to make a profit[36].

Alternatively, the average trade reports a loss when

$$\bar{\Pi}^M_N(\sigma_X,\omega)<0\Rightarrow \bar{I}^M_N(\sigma_X,\omega)<C(\sigma_X) \tag{4.50a}$$

$$\Rightarrow \beta^j_X(t,\omega)<0 \text{ and } \mathrm{sgn}(d\tilde{B}_X(t,\omega))>0, \text{ or} \tag{4.50b}$$

$$\beta^j_X(t,\omega)>0 \text{ and } \mathrm{sgn}(d\tilde{B}_X(t,\omega))<0 \tag{4.50c}$$

So traders incur losses when they misread the market: they short sell when volatility signals are bullish in (4.50b), or buy and hold when volatility signals are bearish in (4.50c). The foregoing strategies are summarized in

[35](Brogaard, 2010, pg. 2) ("HFTs generate around $2.8 billion in gross annual trading proits and on a per $100 traded earn three-fourths of a penny").

[36](Kirilenko et al., 2011, pg. 3) report that on the day of the Flash Crash, May 6, 2010, HFT traded over 1,455,000 [futures] contracts.

Proposition 4.7 (Profits of high frequency traders)**.**

Let $\bar{I}_N^M(\sigma_X, \omega)$ be the average capital gain per trade for M trades, in a market with N high frequency traders, over a time interval $a < t < b$, for given hedge factor volatility σ_X. Let $C(\sigma_X)$ be the unit cost of a trade. Let $\bar{\Pi}_N^M(\sigma_X, \omega) = \bar{I}_N^M(\sigma_X, \omega) - C(\sigma_X)$ be the average profit per trade, and $\beta_X(t, \omega)$ be exposure to hedge factor X. Then $\bar{\Pi}_N^M(\sigma_X, \omega) \gtrless 0$ according as volatility signals or market sentiment is measured by the direction $\beta_X(t, \omega) sgn(d\tilde{B}_X(t, \omega) \gtrless 0$ for the strategies in (4.45) *and* (4.50)*; and cost and volatility structure in* (4.48) *and* (4.49).

□

Remark 4.4. This proposition underscores the importance of CBOE VIX signals, the so called investor fear gauge[37] for market sentiment for comparatively low trade frequency.

4.5.1 A case study of Nanex charts for high frequency trading in natural gas futures on NYMEX June 8, 2011

This subsection applies our theory to some graphics of high frequency trading in natural gas index futures on June 8, 2011 according to Hunsader (2011). From the outset we note that the quantity $\sigma_X^{-\frac{1}{k}}$ in (4.36) is a measure of precision about the hedge factor[38]. That is, smaller hedge factor volatility implies greater precision, i.e. more information, and larger amplitude, i.e. $\sigma_X \downarrow \Rightarrow \hat{I}^j(\sigma_X, \omega) \uparrow$. The increasing amplitude depicted by the Nanex graphics for the NYMEX Natural Gas index

[37] Whaley (2000).

[38] See (DeGroot, 1970, pg. 38) for definition.

futures (NG), i.e. hedge factor, in Figure 1 and Figure 2 suggest that traders were using information against market makers, i.e. the adverse selection effect was in play. The harmonic bid-ask spread proxies for capital gains from trade represented by $\hat{I}^j(\sigma_X, \omega)$ in (4.36). Figure 3 depicts phase locked short sell price strategies, i.e. a sell off, and natural gas prices plunge. Market makers subsequently tightened the spread, and the volume of trade decreased in Figure 4. However, marketmakers were forced to buy at the high end, and take a loss as depicted in Figure 5. The text of the annotated graphics for FIGURES 6 to 10, as described by Nanex reads:

> The following charts show trade, trade volume, and depth of book prices and relative sizes for the July 2011 Natural Gas futures trading on NYMEX. Depth of book data is color coded by color of the rainbow (ROYGBIV), with red representing high bid/ask size and violet representing low bid/ask size. In this way, we can easily see changes in size to the depth of the trading book for this contract.
>
> Depth of book is 10 levels of bid prices and 10 levels of ask prices. The bid levels start with the best (highest) bid, and drop in price 10 levels. Ask levels start with the best (lowest) ask, and increase in price 10 levels. The different in price between levels is not always the same. It depends on traders submitting bids and offers. In other words, depth of book shows the 10 best bid prices, and 10 best ask prices.
>
> In a normal market, prices move lower when the number of contracts at the top level bid are executed. The next highest bid level then becomes the top level, and the 3rd level becomes the second and so forth. A new level is then added below the previous lowest level. On our our depth charts display, you would see this behavior as a change in color of the top level bid from the red end of the spectrum towards the violet end.
>
> On June 8, 2011, starting at 19:39 Eastern Time, trade prices began oscillating almost harmonically along with the depth of book. However, prices rose as bid were executed, and prices declined when offers were

executed – the exact opposite of a market based on supply and demand. Notice that when the prices go up, the color on the ask side remains mostly unchanged, but the color on the bid side goes from red to violet. When prices go down, the color on the bid side remains mostly unchanged, but the color on the ask side goes from red to violet. This is highly unusual.

Upon closer inspection, we find that price oscillates from low to high when trades are executing against the highest bid price level. After reaching a peak, prices then move down as trades execute against the highest ask price level. This is completely opposite of normal market behavior.

The amplitude (difference between the highest price and lowest price) of each oscillation slowly increases, until a final violent downward swing on high volume. There also appears to be 3 groups of these oscillations or perhaps two intervals separating these oscillations. It's almost as if someone is executing a new algorithm that has it's buying/selling signals crossed. Most disturbing to us is the high volume violent sell off that affects not only the natural gas market, but all the other trading instruments related to it.

5 Conclusion

This paper synthesized the continuous time asset pricing models in Cadogan (2011) (trade strategy representation theorem), and Jarrow and Protter (2010) (K-factor model for portfolio alpha), to produce a simple stock price formula that captures several empirical regularities of stock price dynamics attributable to high frequency trading–according to emergent literature. The model presented here does not address fundamental valuation of stock prices. Nor does it explain what causes a high frequency trader to decide to sell [or buy] in periods of uncertainty.

However, recent quantum cognition theories show that subjects emit a behavioural quantum wave in decision making when faced with uncertainty. Thus, further research in that direction may help explain underlying trade cycles. Additionally, the double exponential representation of HFT stock prices suggests that some kind of exponential heteroskedasticity correction factor, i.e. EGARCH, should be used for performance evaluation of active portfolio management. And that HFT profitability depends on trader ability to predict market volatility by and through market sentiment factors like CBOE VIX for low frequency trade, and ability to predict volatility induced by the bid-ask bounce for ultrahigh frequency trades. The stock price formula also has independently important econometric implications for cost of carry models popularized in the futures pricing literature, and suggests future research in that direction.

6 Appendix: Nanex graphics for NYMEX natural gas index futures

Figure 1: Monotone increasing amplitude of oscillatory integrals for capital gains $\hat{I}_N^M(\sigma_X)$

Nanex

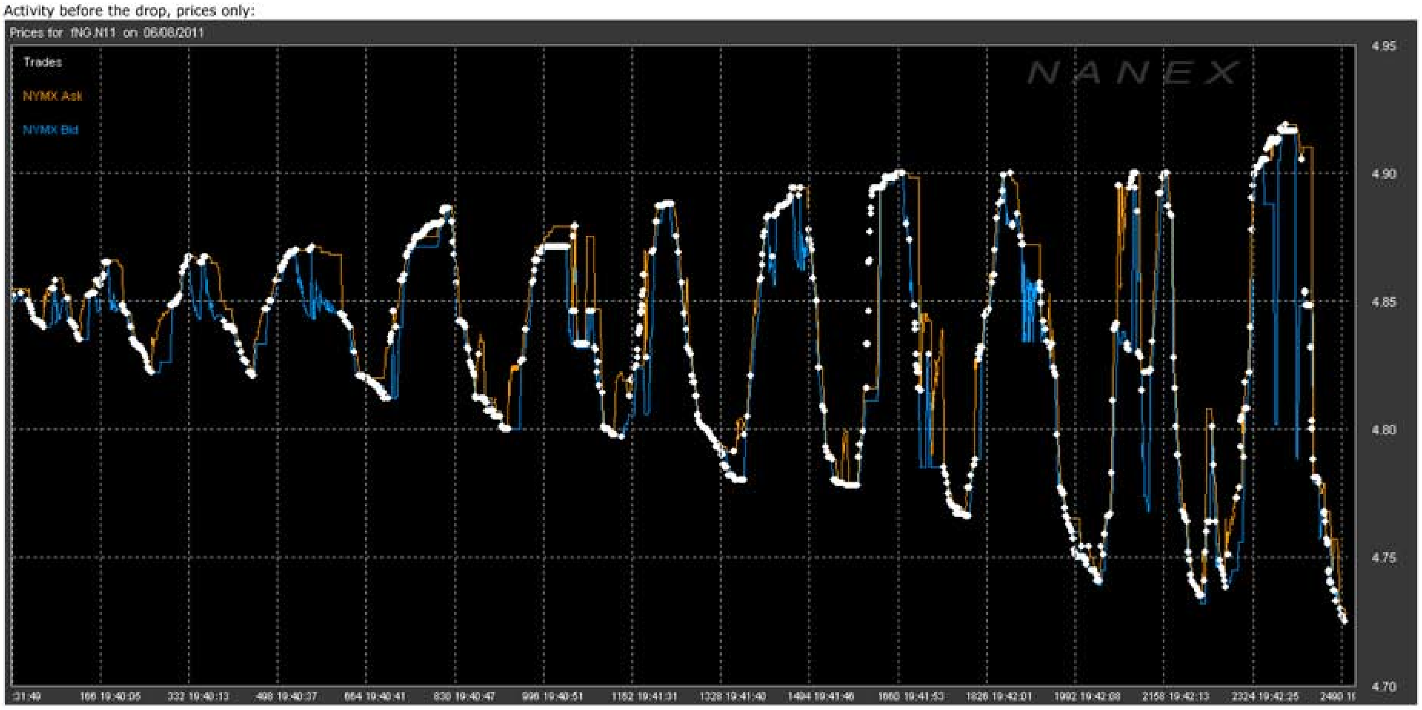
Home Products Testimonials Support Tools Company

Strange Days June 8'th, 2011 - NatGas Algo

Back to Table Of Contents

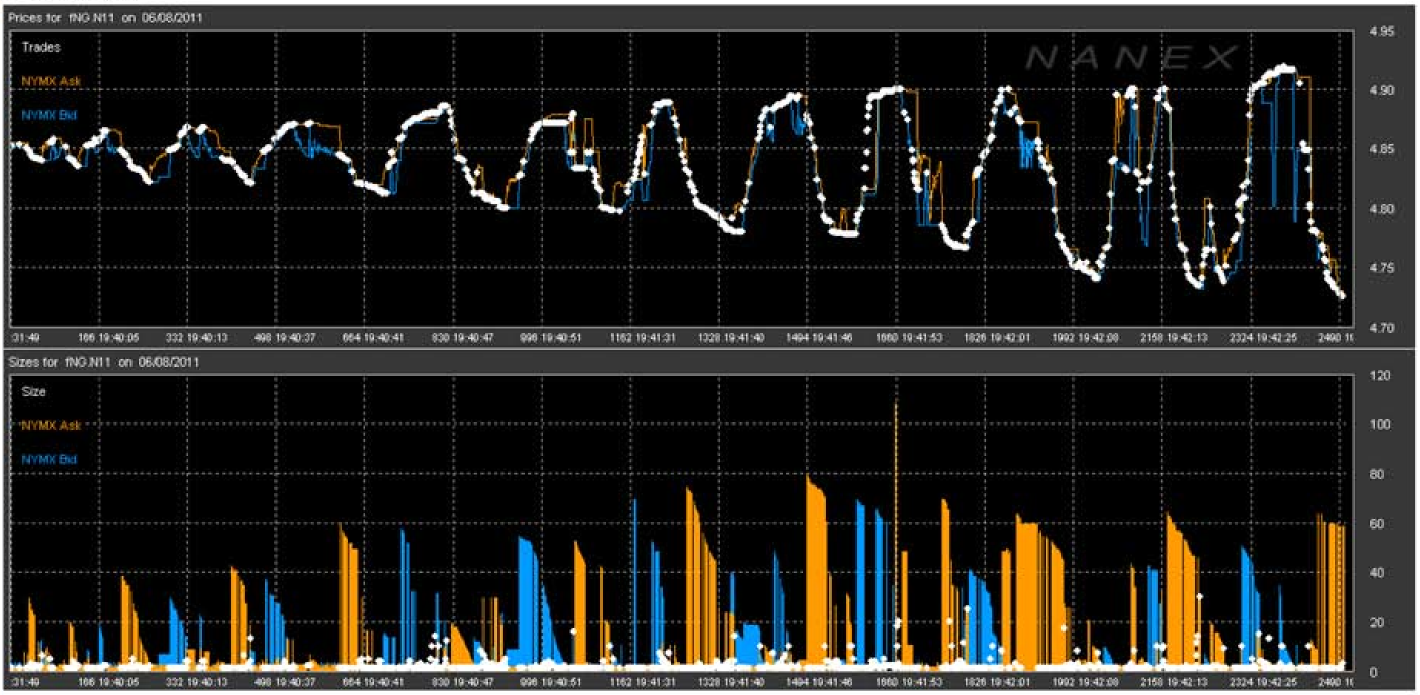
In the following chart, trades and quotes are plotted sequentially as they occur. As such, no data is lost. Exchange's bid and ask prices are colored according to the legend on the left. Trades are colored in white.

Activity before the drop, prices only:



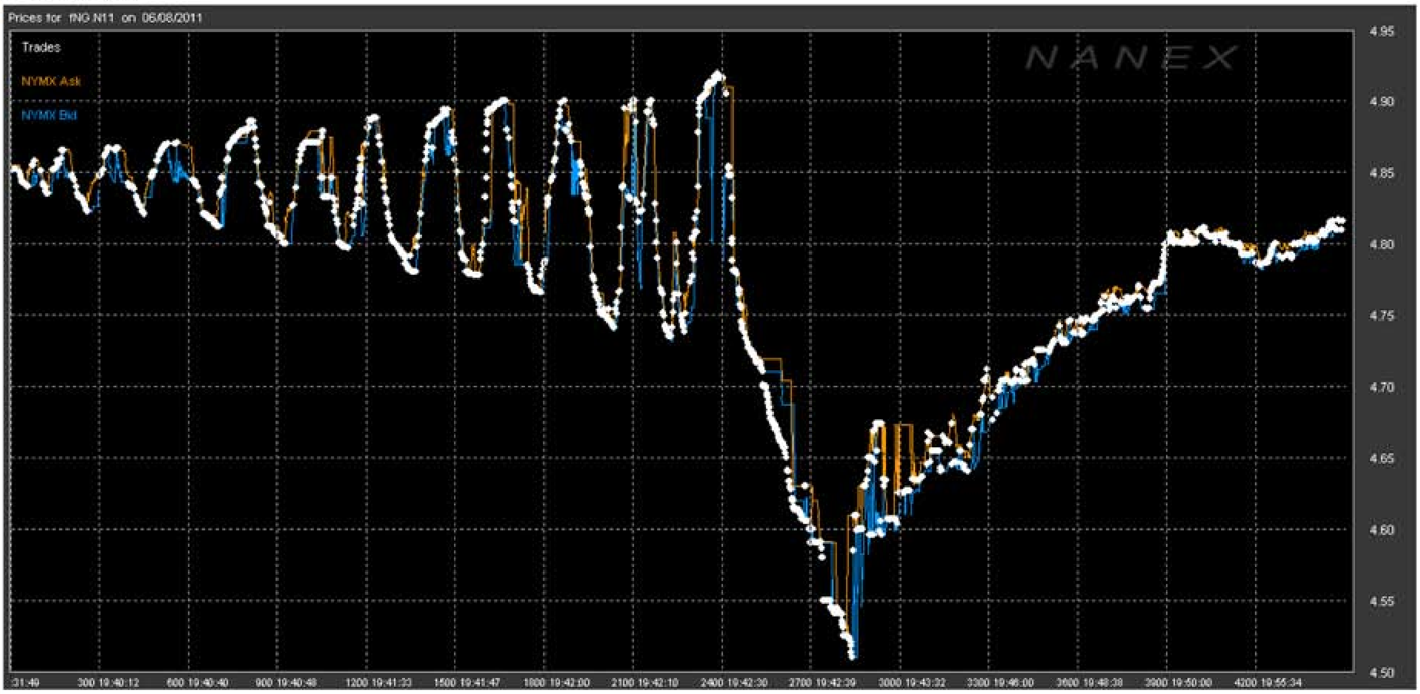
Activity before the drop, prices and size:

Figure 2: Comonotone bid-ask spread and trade size



Activity after the drop, prices only:

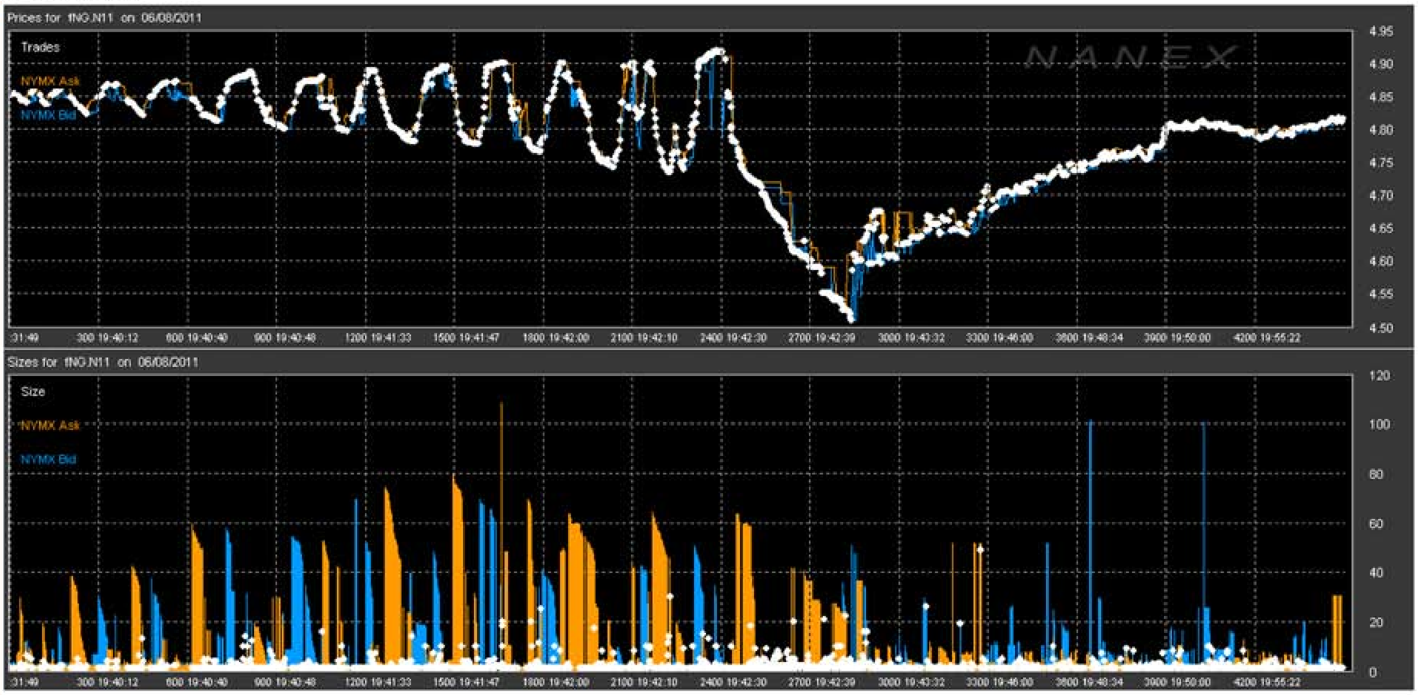
Figure 3: Price plunge from phase locked short sell strategy



Activity after the drop, prices and size:

Figure 4: Marketmaker response to phased lock short sell: Decreased bid-ask spread and trade size

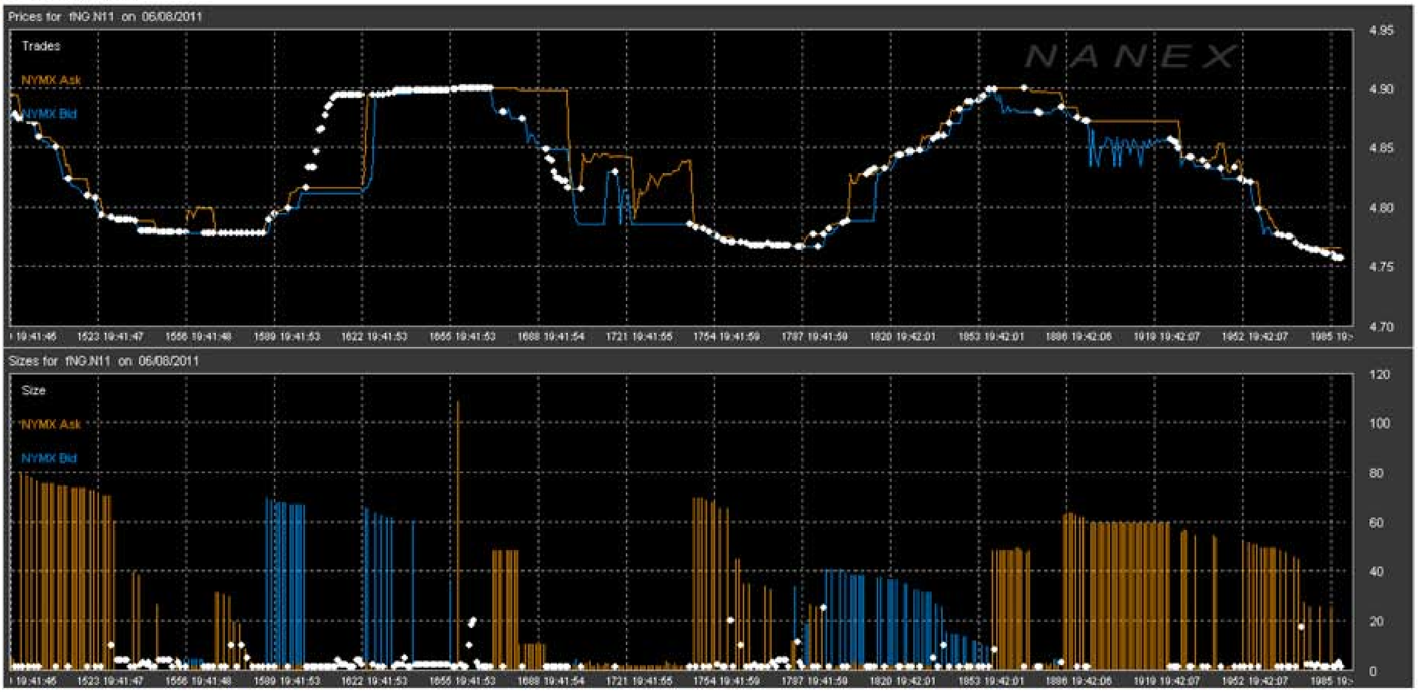
Nanex - Strange Days June 8'th, 2011 - NatGas Algo



Zoom in of the pattern:

http://www.nanex.net/StrangeDays/06082011.html[1/15/2012 12:49:18 PM]

Figure 5: Adverse selection: Marketmaker buys at the high and takes loss



The following charts show trade, trade volume, and depth of book prices and relative sizes for the July 2011 Natural Gas futures trading on NYMEX. Depth of book data is color coded by color of the rainbow (ROYGBIV), with red representing high bid/ask size and violet representing low bid/ask size. In this way, we can easily see changes in size to the depth of the trading book for this contract.

Depth of book is 10 levels of bid prices and 10 levels of ask prices. The bid levels start with the best (highest) bid, and drop in price 10 levels. Ask levels start with the best (lowest) ask, and increase in price 10 levels. The different in price between levels is not always the same. It depends on traders submitting bids and offers. In other words, depth of book shows the 10 best bid prices, and 10 best ask prices.

In a normal market, prices move lower when the number of contracts at the top level bid are executed. The next highest bid level then becomes the top level, and the 3rd level becomes the second and so forth. A new level is then added below the previous lowest level. On our our depth charts display, you would see this behavior as a change in color of the top level bid from the red end of the spectrum towards the violet end.

On June 8, 2011, starting at 19:39 Eastern Time, trade prices began oscillating almost harmonically along with the depth of book. However, prices rose as bid were executed, and prices declined when offers were executed -- the exact opposite of a market based on supply and demand. Notice that when the prices go up, the color on the ask side remains mostly unchanged, but the color on the bid side goes from red to violet. When prices go down, the color on the bid side remains mostly unchanged, but the color on the ask side goes from red to violet. This is highly unusual.

Upon closer inspection, we find that price oscillates from low to high when trades are executing against the highest bid price level. After reaching a peak, prices then move down as trades execute against the highest ask price level. This is completely opposite of normal market behavior.

The amplitude (difference between the highest price and lowest price) of each oscillation slowly increases, until a final violent downward swing on high volume. There also appears to be 3 groups of these oscillations or perhaps two intervals separating these oscillations. It's almost as if someone is executing a new algorithm that has it's buying/selling signals crossed. Most disturbing to us is the high volume violent sell off that affects not only the natural gas market, but all the other trading instruments related to it.

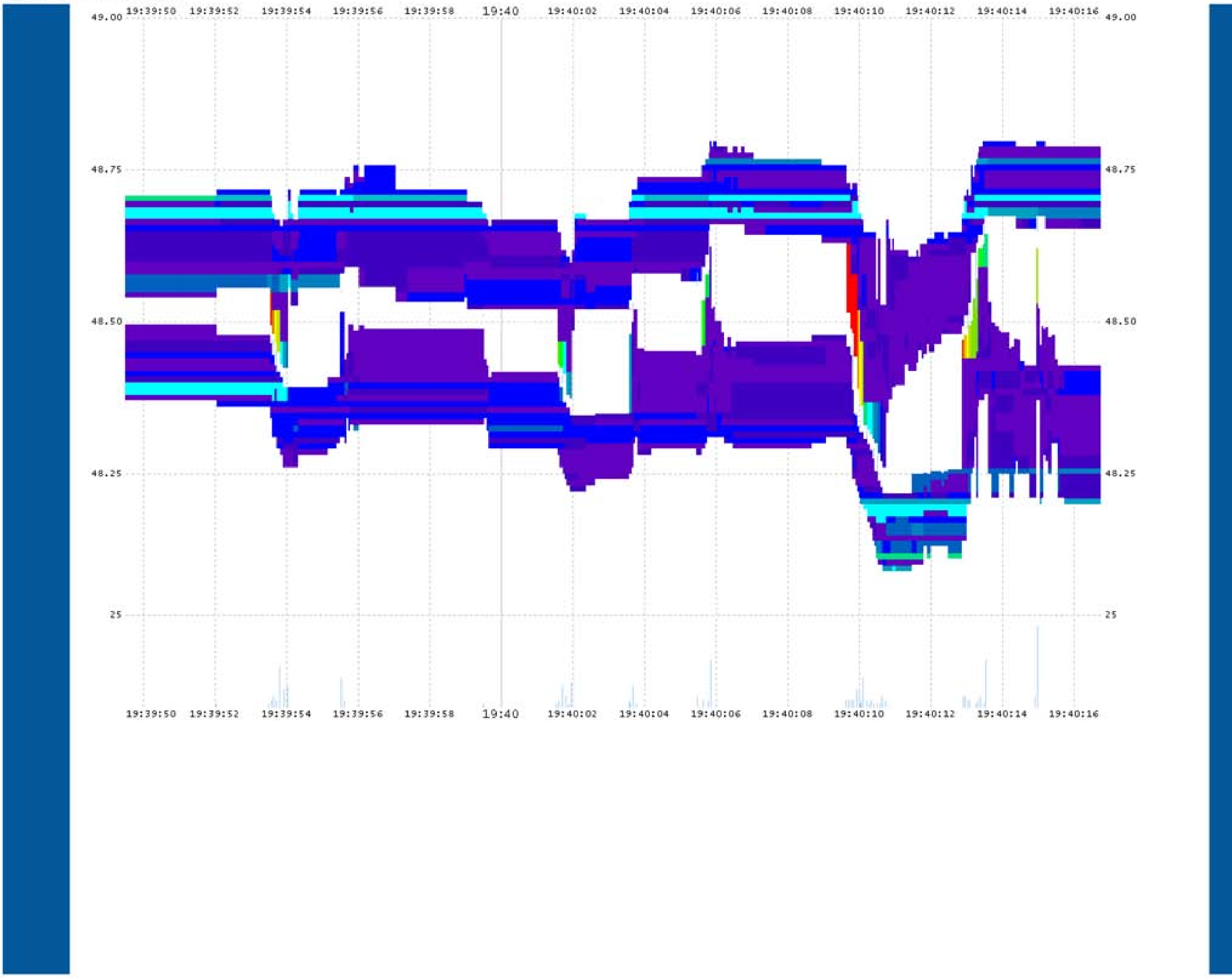
Figure 6:

Nanex - Strange Days June 8'th, 2011 - NatGas Algo

http://www.nanex.net/StrangeDays/06082011.html[1/15/2012 12:49:18 PM]

Figure 7:

Nanex - Strange Days June 8'th, 2011 - NatGas Algo



http://www.nanex.net/StrangeDays/06082011.html[1/15/2012 12:49:18 PM]

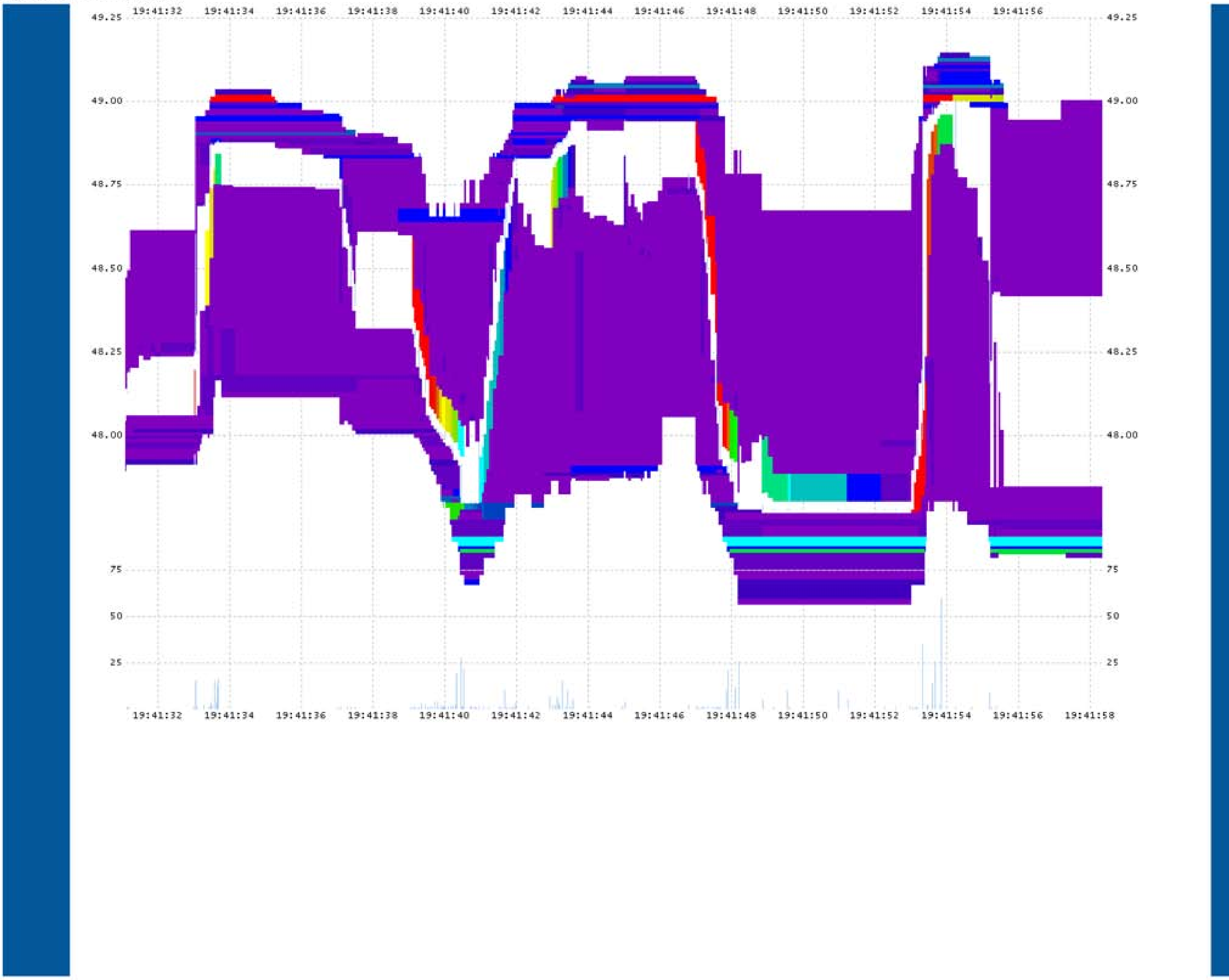
Figure 8:

Nanex - Strange Days June 8'th, 2011 - NatGas Algo

http://www.nanex.net/StrangeDays/06082011.html[1/15/2012 12:49:18 PM]

Figure 9:

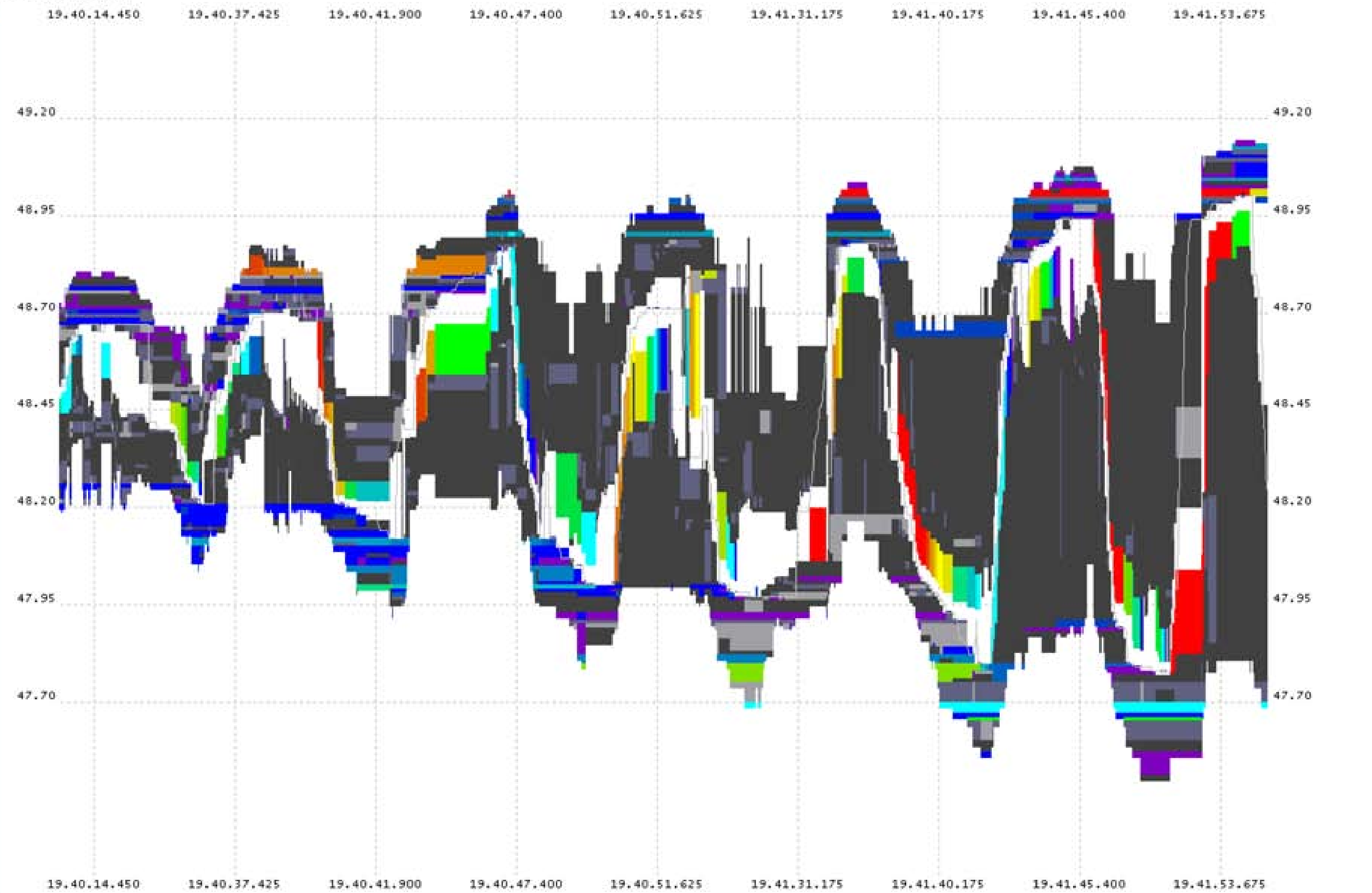
Nanex - Strange Days June 8'th, 2011 - NatGas Algo



http://www.nanex.net/StrangeDays/06082011.html[1/15/2012 12:49:18 PM]

Figure 10:

Nanex - Strange Days June 8'th, 2011 - NatGas Algo



Inquiries: pr@nanex.net

Publication Date: 06/08/2011
http://www.nanex.net

References

Admati, A. and P. Pfleiderer (1988). A Theory of Intraday Patterns: Volume and Price Variability. *Review of Financial Studies 1*(1), 3–40.

Akerlof, G. A. (1970, Aug.). The Market for ”Lemons”: Quality Uncertainty and the Market Mechanism. *Quarterly Journal of Economics 84*(3), 488–500.

Aldridge, I. (2011, Sept.). How fast Can You Really Trade? *Futures Magazine*.

Avellaneda, M. and J.-H. Lee (2010). Statistical arbitrage in the US equities Market. *Quantitative Finance 10*(7), 761–782.

Brogaard, J. A. (2010, Nov.). High Frequency Trading and Its Impact On Market Quality. Working Paper, Kellog School of Management, Department of Finance, Northwestern University. Available at SSRN eLibrary http://ssrn.com/abstract=1641387.

Busse, J. A. (1999). Volatility Timing in Mutual Funds: Evidence From Daily Returns. *Review of Financial Studies 12*(5), 1009–1041.

Cadogan, G. (2011). Alpha Representation For Active Portfolio Management and High Frequency Trading In Seemingly Efficient Markets. In *Proceedings of Joint Statistical Meeting (JSM)*, Volume CD-ROM, Alexandria, VA, pp. 673–687. Business and Economic Statistics Section: American Statistical Association.

Campbell, J. Y., A. W. Lo, and A. C. MacKinlay (1997). *The Econometrics of Financial Markets*. Princeton, NJ: Princeton University Press.

Christensen, K., R. C. Oomen, and M. Podolskij (2011, May). Fact or Friction: Jumps at Ultra High Frequency. *SSRN eLibrary*. Available at http://ssrn.com/paper=1848774.

Christopherson, J. A., W. A. Ferson, and D. A. Glassman (1998, Spring). Conditional Manager Alphas On Economic Information: Another Look At The Persistence Of Performance. *Review of Financial Studies 11*(1), 111–142.

de Long, B., A. Shleifer, L. H. Summers, and R. J. Waldman (1990, June). Positive Feedback Investment Strategies and Destabilizing Rational Speculation. *Journal of Finance 45*(2), 379–395.

DeGroot, M. (1970). *Optimal Statistical Decisions*. New York, N.Y.: McGraw-Hill, Inc.

Doob, J. L. (1949). Heuristic Appproach to The Kolmogorov-Smirnov Theorems. *Ann. Math. Statist. 20*(3), 393–403.

Engle, R. (2004, June). Risk and Volatility: Econometric Models and Financial Practice. *American Economic Review 94*(3), 405–420.

Engle, R. F. and V. K. Ng (1993, Dec.). Measuring and Testing The Impact of News on Volatility. *Journal of Finance 48*(5), 1749–1778.

Fama, E. and K. French (1993). Common Risk Factors in the Return on Bonds and Stocks. *Journal of Financial Economics 33*(1), 3–53.

Gordon, M. (1959, May). Dividends, Earnings, and Stock Prices. *Review of Economics and Statistics 41*(2), 99–105.

Grinold, R. C. and R. N. Kahn (2000). *Active Portfolio Management: A Quantitative Approach for Providing Superior Returns and Controlling Risk* (2nd ed.). New York: McGraw-Hill, Inc.

Hull, J. (2006). *Options, Futures, and Other Derivatives* (6th ed.). New Jersey: Prentice-Hall, Inc.

Hunsader, E. S. (2011, August). NANEX Graphics for NYNEX Natural Gas Index Futures June 8, 2011. Webpage. Available at http://www.nanex.net/StrangeDays/06082011.html.

Infantino, L. R. and S. Itzhaki (2010, June). Developing High Frequency Equities Trading Models. Master's thesis, MIT Sloan, Cambridge, MA. Available at http://dspace.mit.edu/handle/1721.1/59122.

Jarrow, R. and P. Protter (2010, April). Positive Alphas, Abnormal Performance, and Illusionary Arbitrage. Johnson School Research Paper #19-2010, Dept. Finance, Cornell Univ. Available at http://ssrn.com/abstract=1593051. Forthcoming, *Mathematical Finance*.

Jarrow, R. A. (2010, Summer). Active Portfolio Manageement and Positiive Alphas: Fact or fantasy? *Journal of Portfolio Management 36*(4), 17–22.

Jarrow, R. A. and P. Protter (2011). A Dysfunctional Role of High Frequency Trading in Electronic Markets. *SSRN eLibrary*. Available at http://ssrn.com/paper=1781124.

Joint Advisory Committee on Emerging Regulatory Issues (2011, Feb.). Recommendations Regarding Regulatory Responses to the Market Events of May 6, 2010: Summary Report of the Joint CFTC-SEC Advisory Committee on Emerging Regulatory Issues (Feb. 18, 2011). *Joint CFTC-SEC Advisory Committee on Emerging Regulatory Issues*. FINRA Conference May 23, 2011. Available at http://www.sec.gov/spotlight/sec-cftcjointcommittee/021811-report.pdf.

Kaminska, I. (2011, August). ”HFT is killing the emini,” says Nanex. *Financial Times*. Available at http://ftalphaville.ft.com/blog/2011/08/08/646276/hft-is-killing-the-emini-says-nanex/.

Karatzas, I. and S. E. Shreve (1991). *Brownian Motion and Stochastic Calculus* (2nd ed.). Graduate Text in Mathematics. New York, N. Y.: Springer-Verlag.

Karlin, S. and H. M. Taylor (1981). *A Second Course in Stochastic Processes*. New York, NY: Academic Press, Inc.

Kirilenko, A. A., A. P. S. Kyle, M. Samadi, and T. Tuzun (2011, May). The Flash Crash: The Impact of High Frequency Trading on an Electronic Market. *SSRN eLibrary*. Available at http://ssrn.com/abstract=1686004.

MacKinlay, A. and K. Ramaswamy (1988). Index-futures arbitrage and the behavior of stock index futures prices. *Review of Financial Studies 1*(2), 137–158.

Madhavan, A. (2011). Exchange-Traded Funds, Market Structure and the Flash Crash. *SSRN eLibrary*. Available at http://ssrn.com/paper=1932925.

McTague, J. (2010, Aug.). Was The Flash Crash Rigged? *Barron’s*. Available at http://online.barrons.com/article/SB50001424052970204304404575449930920336058.html

Nelson, D. (1991, March). Conditional Heteroskedasticity in Asset Pricing: A New Approach. *Econommetrica 59*(2), 347–370.

Noehel, T., Z. J. Wang, and J. Zheng (2010). Side-by-Side Management of Hedge Funds and Mutual Funds. *Review of Financial Studies 23*(6), 2342–2373.

Øksendal, B. (2003). *Stochastic Differential Equations: An IntroductionWith Applications* (6th ed.). Universitext. New York: Springer-Verlag.

Shephard, N. (1996). *Time Series Models in Econometrics, Finance and Other Fields*, Chapter Statistical Aspects of ARCH and Stochastic Volatility, pp. 1–67. London: Chapman & Hall.

Shiller, R. (1989). Comovement in Stock Prices and Comovement in Dividends. *Journal of Finance 44*(2), 719–729.

Shiller, R. J. and A. E. Beltratti (1992). Stock prices and bond yields: Can their comovements be explained in terms of present value models? *Journal of Monetary Economics 30*(1), 25 – 46.

Sornette, D. and S. Von der Becke (2011). Crashes and High Frequency Trading. *SSRN eLibrary*. Available at http://ssrn.com/paper=1976249.

Stein, E. M. (1993). *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton, NJ: Princeton Univ. Press.

Stoll, H. R. and R. E. Whaley (1990). The dynamics of stock index and stock index futures returns. *Journal of Financial and Quantitative Analysis 25*(04), 441–468.

Tobin, J. (1984, Fall). A mean variaance approach to fundamental valuation. *Journal of Portfolio Management*, 26–32.

Whaley, R. E. (2000). The Investor Fear Gauge: Explication of the CBOE VIX. *Journal of Portfolio Management 26*(3), 12–17.

Whitten, D. (2011, August 17). Has High Frequency Trading, Futures, Derivatives, etc., Rigged The Game Against Individual Investors? Not Really. iStockAnalyst.com. Available at http://www.istockanalyst.com/finance/story/5364511/has-high-frequency-trading-futures-derivatives-etc-rigged-the-game-against-individual-investors-not-really.

Zhang, F. (2010). High-Frequency Trading, Stock Volatility, and Price Discovery. *SSRN eLibrary*. Available at http://ssrn.com/paper=1691679.

Zhang, N. (2011, June). Properties of the SABR Model. Working Paper, Dep't of Mathematics, Uppasala Uviv. Available at http:// uu.diva-portal.org/smash/get/diva2:430537/FULLTEXT01.